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Non-standard construction of Hamiltonian structures

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Abstract. Examples of the construction of Hamiltonian structures for dynamical systems in field theory (including one reputedly non-Hamiltonian problem) without using Lagrangians, are presented. The recently developed method used requires the knowledge of one constant of motion of the system under consideration and one solution of the symmetry equation.

1. Introduction

Hamiltonian theories have been widely used in almost all areas of physics. The usual approach consists of constructing the momenta (canonically conjugate to the coordinates) and the Hamiltonian starting from a Lagrangian formulation of the system under consideration. Nevertheless, there has been increasing interest in studying non-standard procedures to produce Hamiltonian structures starting from the equations of motion only, without using a Lagrangian [1–9]. These approaches usually deal with systems which are naturally described in terms of non-canonical variables and the Hamiltonian formulation is consequently written using non-canonical Poisson matrices, which in many cases are singular. Due to the lack of a general procedure to generate Hamiltonian structures from scratch, in most instances the results have been obtained by extremely inspired guesswork.

The purpose of this paper is to present examples of the application of a newly devised method [10, 11] which constitutes a *general technique* for the construction of Hamiltonian structures for dynamical systems. We discuss systems described by nonlinear equations in field theory and also some linear equations such as the time-dependent Schrödinger equations and the heat equation, which according to the usual belief, is clearly non-Hamiltonian [5].

In section 2 we give a brief outline of the method, the rest of the paper is devoted to the construction of the examples.

2. Brief outline of the method

In this section we partially summarize the results of [10]. We use classical mechanical notation nevertheless, the results may be easily applied to field theory as well, as we will see at the end of this section.

Consider a dynamical system defined by equations which have been cast in first-order form,

$$\frac{\mathrm{d}x^a}{\mathrm{d}t} = f^a(x^b) \qquad a, b = 1, \ldots, N. \tag{1}$$

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A Hamiltonian structure for it consists of an antisymmetric matrix, $J^{ab}(x^c)$ and a Hamiltonian $H(x^c)$ such that J^{ab} is the Poisson bracket for the variables x^a and x^b (which are non-canonical in general) and H is the Hamiltonian for system (1). In addition to its antisymmetry, the matrix J^{ab} is required to satisfy the Jacobi identity and to reproduce, in conjunction with the Hamiltonian H, the dynamical equations (1), i.e.

$$J^{ab}{}_{,d}J^{dc} + J^{bc}{}_{,d}J^{da} + J^{ca}{}_{,d}J^{db} \equiv 0 \tag{2}$$

and,

$$J^{ab}\frac{\partial H}{\partial x^b} = f^a. \tag{3}$$

It has been proved [10] that one solution to the problem of finding a Hamiltonian structure for a given dynamical system is provided by one constant of motion which may be used as the Hamiltonian H, and a symmetry vector η^a which allows the construction of a Poisson matrix J^{ab}. The constant of motion and the symmetry vector satisfy,

$$\mathcal{L}_f H = 0 \tag{4}$$

$$(\partial_t + \mathcal{L}_f)\eta^a = 0 \tag{5}$$

respectively, where $\mathcal{L}_f$ is the Lie derivative along f (for a definition, see [12], for instance). In addition, it is required that the deformation K of H along η^a,

$$K \equiv \frac{\partial H}{\partial x^a}\eta^a = \mathcal{L}_\eta H \tag{6}$$

is non-vanishing. The Poisson matrix J^{ab} is constructed as the antisymmetrized product of the flow vector f^a and the 'normalized' symmetry vector η^b/K,

$$J^{ab} = \frac{1}{K}(f^a\eta^b - f^b\eta^a). \tag{7}$$

The Poisson matrix so constructed has rank 2 and it is, therefore, singular in many instances. Adding together two Poisson matrices constructed according to (7) will not increase its rank. It will just redefine the symmetry vector used to construct it. One method to increase the rank of such a Poisson matrix is presented in [10].

Let us now deal with systems with infinitely many degrees of freedom defined by some field $\phi(x,t)$, where x denotes the coordinates of a point in space. The dynamical equation (1) is now,

$$\dot{\phi}(x,t) = F[\phi, x] \tag{8}$$

where F is a functional of ϕ for every point x in space. All the above discussion remains valid replacing x^a by $\phi(x)$, tensors $T^{ab\ldots c}(x^a)$ by functionals Θ which depend on some spatial coordinates, $\Theta[\phi, x, y, \ldots, z]$ and partial derivatives $\partial/\partial x^a$ by functional derivatives $\delta/\delta\phi(x)$. Details are given in [3].

3. The heat equation

Let us consider the heat equation,

$$u_t = u_{xx}. \tag{9}$$

It is easy to see that

$$\eta(x) = \Delta \tag{10}$$

is a symmetry transformation for it, where ϵ is any real number.

If we assume periodic boundary conditions in $x \in [-a/2, a/2]$ then the quantity

$$H = \int_{-a/2}^{a/2} u \, \mathrm{d}x \tag{11}$$

is conserved.

The deformation of H along η is given by

$$\mathcal{L}_\eta H = \int_{-a/2}^{a/2} \mathrm{d}x \, \frac{\delta H}{\delta u(x)} \eta(x) = \Delta \int_{-a/2}^{a/2} \mathrm{d}x = \Delta a \tag{12}$$

where all the integrals are taken from $-a/2$ to $a/2$.

We can now construct a Poisson matrix associated with the Hamiltonian (11) for equation (9),

$$J(x, y) = \frac{u_{xx}(x) - u_{yy}(y)}{a}. \tag{13}$$

Let us explicitly show that J^{ab} together with H provide a Hamiltonian formulation for the heat equation. In fact,

$$\begin{aligned}
[u(x), H] &= \int_{-a/2}^{a/2} \mathrm{d}y \, J(x, y) \frac{\delta H}{\delta u(y)} \\
&= \frac{1}{a} u_{xx}(x) \int_{-a/2}^{a/2} \mathrm{d}y - \frac{1}{a} \int_{-a/2}^{a/2} u_{yy}(y) \, \mathrm{d}y \\
&= u_{xx}(x).
\end{aligned}$$

So we see that,

$$\dot{u} = [u, H] \tag{14}$$

as required. It is worth noting that according to folk tradition, this equation cannot be endowed with a Hamiltonian structure. In Salmon's words [5]: 'By anyone's definition, (0.1) is non-Hamiltonian'. In his paper, (0.1) is the heat equation subject to periodic boundary conditions.

4. Time-dependent Schrödinger equations

Consider the following equations of motion

$$\psi_t = \mathrm{i}(\psi_{xx} + V(x, t)\psi) \equiv f \tag{15}$$

and its complex conjugate

$$\psi^*{}_t = -\mathrm{i}(\psi^*{}_{xx} + V(x, t)\psi^*) \equiv f^*. \tag{16}$$

Note that we discuss the case of a time-dependent potential. We use one-dimensional notation for simplicity only, our results hold irrespective of the dimensionality of space.

It is a straightforward matter to realize that multiplication of the variables ψ and ψ^* by two different constants, $(1 + \lambda)$ and $(1 + \mu\lambda)$ respectively, constitutes a symmetry transformation for the Schrödinger equations (15) and (16). The infinitesimal version of this transformation is

$$\eta = \psi \tag{17}$$

and

$$\eta^* = \mu\psi^*. \tag{18}$$

The usual probability conservation statement means that

$$H = \int \psi^*(x)\psi(x)\,\mathrm{d}x \tag{19}$$

is a conserved quantity as it may be easily proved. Its deformation K along the symmetry vector defined by (17) and (18) may be written in terms of variational derivatives as

$$K \equiv \int \frac{\delta H}{\delta\psi(x)}\eta(x)\,\mathrm{d}x + \int \frac{\delta H}{\delta\psi^*(x)}\eta^*(x)\,\mathrm{d}x \tag{20}$$

and a straightforward calculation yields

$$K = (1+\mu)H \neq 0 \tag{21}$$

for any μ such that $1+\mu \neq 0$. The Poisson structure is defined by

$$[\psi(x), \psi(y)] = \frac{1}{K}(f(x)\eta(y) - \eta(x)f(y)) \tag{22}$$

$$[\psi(x), \psi^*(y)] = \frac{1}{K}(f(x)\eta^*(y) - \eta(x)\ f^*(y)) \tag{23}$$

and

$$[\psi^*(x), \psi^*(y)] = \frac{1}{K}(f^*(x)\eta^*(y) - \eta^*(x)f^*(y)). \tag{24}$$

We have thus constructed a family of Poisson structures which depend on the parameter μ for a given Hamiltonian H. Note that despite the time-dependent character of the Schrödinger equations (15) and (16) the Hamiltonian (19) is conserved, while the usual Hamiltonian contains the time-dependent potential $V(x,t)$ and it is not conserved. On the other hand, our Poisson structures are time dependent. This is, as far as we know, a novel feature for Poisson matrices, which means that, even in the case of a regular matrix, the Hamiltonian structure is not derivable from a Lagrangian.

5. The Korteweg–de Vries equation

The equation of motion is

$$u_t = -uu_x - u_{xxx} \equiv f. \tag{25}$$

It is not difficult to see that the symmetry transformation for it is given by [10]

$$\eta = (-2u - xu_x + 3t(uu_x + u_{xxx})). \tag{26}$$

In fact, to prove that (26) is a symmetry transformation for the KdV equation, it is enough to check that η satisfies the symmetry equation (5) with f defined by (25).

To get a Hamiltonian structure for the KdV equation we need to construct constants of motion which are non-trivially deformed by η. In [10] the energy

$$H_1 = \int u^2\,\mathrm{d}x \tag{27}$$

was used as a constant of motion to complete the Hamiltonian structure. Note that the deformation K_1 of H_1 along η is non-vanishing

$$K_1 \equiv \int \frac{\delta H_1}{\delta u(x)}\eta(x)\,\mathrm{d}x = -3H_1. \tag{28}$$

The Poisson structure $J_1(x,y)$ is given by

$$J_1(x,y) = \frac{1}{K_1}(f(x)\eta(y) - f(y)\eta(x)). \tag{29}$$

Consider now H_2

$$H_2 = \int \left(-\frac{{u_x}^2}{2} + u^3 \right) \mathrm{d}x \tag{30}$$

which is also conserved. Its deformation K_2 along η is

$$K_2 \equiv \int \frac{\delta H_2}{\delta u(x)} \eta(x) \, \mathrm{d}x = -5H_2. \tag{31}$$

The Poisson structure $J_2(x, y)$ is given by

$$J_2(x, y) = \frac{1}{K_2} (f(x)\eta(y) - f(y)\eta(x)). \tag{32}$$

We have been able to construct two different Hamiltonian structures based on one symmetry vector η and two conserved quantities H_1 and H_2 as the Hamiltonians of each of the structures. The Poisson structures J_1 and J_2 are built as the antisymmetric product of the evolution vector f and the symmetry vector η normalized using the deformations K_1 and K_2 of H_1 and H_2, respectively. Note that a different but closely related scheme exists in which no normalization of the antisymmetric product is needed provided that the Hamiltonians $-\frac{1}{3}\log H_1$ and $-\frac{1}{5}\log H_2$ are used instead of H_1 and H_2 respectively. In this case the Poisson matrix for both Hamiltonian structures is exactly the same one.

A similar construction may be performed with the rest of the constants of motion which belong to this family, as they appear, for instance, in [13]. We have then constructed a set of infinitely many Hamiltonian structures for the KdV based on one symmetry transformation and different constants of motion. Let us remark that as it was mentioned in section 2, the matrices $J(x, y)$ constructed according to (7) have rank 2, hence, there are many Casimir functions which have vanishing Poisson bracket relations with any other dynamical quantity. A few words regarding the construction of Casimir functions for Poisson structures such as these seem in order. Time-independent Casimir functions are, of course, constants of motion. Consider now a different evolution along another parameter (call it s) which is given in terms of the symmetry vector η^a. In other words, consider

$$\frac{\mathrm{d}x^a}{\mathrm{d}s} = \eta^a(x^b) \qquad a, b = 1, \ldots, N. \tag{33}$$

Deformation of constants of the motion along η^a may be viewed as the evolution of such constants in the parameter s. A Casimir function is such that its evolutions (both in time and in the s parameter) vanish. So, the construction of Casimir functions may be viewed as the search of entities which are simultaneously constant for both the time and s evolutions.

To illustrate this, take for instance equations (28) and (31) and rewrite them as

$$\frac{\mathrm{d}H_1}{\mathrm{d}s} = -3H_1 \tag{34}$$

and

$$\frac{\mathrm{d}H_2}{\mathrm{d}s} = -5H_2 \tag{35}$$

solve them and eliminate the parameter s to get that ${H_1}^{1/3}{H_2}^{-1/5}$ is a Casimir function for both of the Poisson matrices (29) and (32). Similarly, one can get Casimir functions for other Poisson matrices.

6. The Burgers equation

Consider the Burgers equation,

$$u_t = u_{xx} + 2uu_x \equiv f \tag{36}$$

x being in $[-a/2, a/2]$. It is straightforward to prove that

$$H_1 = \int_{-a/2}^{a/2} \mathrm{d}x\, u(x) \tag{37}$$

$$H_2 = \int_{-a/2}^{a/2} \mathrm{d}x\ \exp\left[\int_{-a/2}^{x} u(y)\,\mathrm{d}y\right] \tag{38}$$

are conserved quantities for it, if we assume that the field vanishes at the boundary $\pm a/2$.

A symmetry transformation for equation (36) is given by

$$\eta(x) = u(x)\exp\left[-\int_{-a/2}^{x} u\right]. \tag{39}$$

To see this, it is enough to show that

$$\mathcal{L}_f \eta = \int_{-a/2}^{a/2} \mathrm{d}y\ \left(\frac{\delta f(x)}{\delta u(y)}\eta(y) - \frac{\delta\eta(x)}{\delta u(y)} f(y)\right) = 0.$$

In fact,

$$\begin{aligned}\int_{-a/2}^{a/2} \mathrm{d}y\ \frac{\delta f(x)}{\delta u(y)}\eta(y) &= \int_{-a/2}^{a/2} \mathrm{d}y\ \frac{\delta\eta(x)}{\delta u(y)} f(y)\\ &= \exp\left(\int_{-a/2}^{x} u(y)\,\mathrm{d}y\right)(u_{xx} + uu_x - u^3).\end{aligned}$$

The deformation of H_1 along η does not vanish,

$$\begin{aligned}K_1 = \mathcal{L}_\eta H_1 &= \int_{-a/2}^{a/2} \mathrm{d}x\ \frac{\delta H_1}{\delta u(x)}\eta(x)\\ &= \int_{-a/2}^{a/2} \mathrm{d}x\, u(x)\exp\left(-\int_{-a/2}^{x} \mathrm{d}w\, u(w)\right)\\ &= 1 - \exp(-H_1).\end{aligned}$$

It may be proved that the deformation of H_2 along η is also non-vanishing,

$$\begin{aligned}K_2 = \mathcal{L}_\eta H_2 &= \int_{-a/2}^{a/2} \mathrm{d}x\ \frac{\delta H_2}{\delta u(x)}\eta(x)\\ &= \int_{-a/2}^{a/2} \mathrm{d}z\ \exp\left(\int_{-a/2}^{z}\right)u(z)\left[\int_{z}^{a/2} \mathrm{d}x\ \exp\left(\int_{-a/2}^{x} \mathrm{d}w\, u(w)\right)\right]\\ &= H_2 - a.\end{aligned}$$

Therefore, we may construct Hamiltonian theories for the Burgers equation using either H_1 or H_2 as Hamiltonians. The appropriate Poisson matrices are

$$J_1(x, y) = \frac{f(x)\eta(y) - f(y)\eta(x)}{K_1} \tag{40}$$

and

$$J_2(x, y) = \frac{f(x)\eta(y) - f(y)\eta(x)}{K_2} \tag{41}$$

respectively.

It is easy to check, for example, that,

$$C = \frac{\mathrm{e}^{H_1} - 1}{H_2 - a} \tag{42}$$

is a Casimir for both of the Poisson brackets defined by (40) and (41), where we have used the approach described at the end of the preceding section.

7. The Harry–Dym equation

The Harry–Dym equation is

$$u_t = (u^{-1/2})_{xxx} \equiv f \tag{43}$$

where $x \in [-a/2, a/2]$. If we assume periodic boundary conditions, H_1 and H_2

$$H_1 = \int_{-a/2}^{a/2} u \,\mathrm{d}x \tag{44}$$

$$H_2 = \int_{-a/2}^{a/2} u^{1/2} \,\mathrm{d}x \tag{45}$$

are conserved quantities. Note that a may be set equal to ∞.

Let us define the vector field

$$\xi(x) = Au - Bxu_x \tag{46}$$

where A and B are real constants. Let us now compute the Lie derivative of it along f,

$$\mathcal{L}_f \xi = 3(\tfrac{1}{2}A + B)(u^{-1/2})_{xxx} = 3(\tfrac{1}{2}A + B)f. \tag{47}$$

Therefore, η

$$\eta = -3t(\tfrac{1}{2}A + B)f + \xi \tag{48}$$

is a symmetry transformation and can be used to construct a Hamiltonian theory provided it deforms some Hamiltonian non-trivially.

This is exactly the case for H_1 and H_2. In fact,

$$K_1 \equiv \mathcal{L}_\eta H_1 = (A + B)H_1 \tag{49}$$

$$K_2 \equiv \mathcal{L}_\eta H_2 = \left(\frac{A}{2} + B\right) H_2 \tag{50}$$

so we have one family of Poisson matrices associated with H_1 and another one associated with H_2. They are

$$J_1(x, y) = \frac{(u^{-1/2})_{xxx}(Au(y) - Byu_y) - (u^{-1/2})_{yyy}(Au(x) - Bxu_x)}{K_1} \tag{51}$$

and

$$J_2(x, y) = \frac{(u^{-1/2})_{xxx}(Au(y) - Byu_y) - (u^{-1/2})_{yyy}(Au(x) - Bxu_x)}{K_2} \tag{52}$$

respectively. Of course, we must be careful to choose A and B in such a way that either K_1 or K_2 (or both) is non-vanishing.

8. Conclusions

We have constructed Hamiltonian structures, without using Lagrangians, for several linear and nonlinear systems of partial differential equations (field theory) based on a recently devised method [10, 11], which needs only of the knowledge of a constant of motion and a solution of the symmetry equation. The examples include the heat equation which has, up to now, been considered to be non-Hamiltonian [5]. The structures found are singular in the sense that there exist Casimir functions (which have vanishing Poisson bracket with any dynamical variable). This feature is present in many other Hamiltonian theories, and it is sometimes unavoidable (as it is in the case of gauge and constrained systems), and it also appears in Hamiltonian descriptions of some fluids. We should stress that being able to produce a Hamiltonian structure (albeit singular) for a system of differential equations constitutes progress with respect to the situation of having no Hamiltonian structure at all. More examples will be discussed in forthcoming articles.

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References

[1] Morrison P J and Greene J M 1980 *Phys. Rev. Lett.* **45** 790
Morrison P J and Greene J M 1982 *Phys. Rev. Lett.* **48** 569
[2] Morrison P J 1994 Hamiltonian description of the ideal fluid *Proc. 1993 Summer Study Program in Geophysical Fluid Dynamics (Woods Hole Oceanographic Institution Report, WHOI 94)*
[3] Shepherd T G 1990 *Adv. Geophys.* **32** 287
[4] Littlejohn R G 1982 *AIP Conf. Proc.* **28** 47
[5] Salmon R 1988 *Ann. Rev. Fluid Mech.* **20** 225
[6] Rebbi C and Soliani G 1984 *Solitons and Particles* (Singapore: World Scientific)
[7] Hojman S A, Núñez D and Ryan M P Jr 1992 *Phys. Rev.* D **45** 3523
[8] Hojman S A, Núñez D and Ryan M P Jr 1992 *Proc. 6th Marcel Grossmann Meeting (Kyoto)* part A, ed H Sato and T Nakamura (Singapore: World Scientific) p 718
[9] Ryan M P Jr and Hojman S 1993 Directions in general relativity *Proc. 1993 Int. Symp. (Maryland, Papers in Honor of Charles Misner)* vol 1, ed B L Hu, M P Ryan Jr and C V Vishveshwara (Cambridge: Cambridge University Press) p 300
[10] Hojman S A 1996 *J. Phys. A: Math. Gen.* **29** 667
[11] Hojman S A 1996 *Instabilities and Non-equilibrium Structures* vol V, ed E Tirapegui and W Zeller (Dordrecht: Kluwer Academic) p 281
[12] Weinberg S 1975 *Gravitation and Cosmology* (New York: Wiley)
[13] Lax P D 1968 *Comment. Pure Appl. Math.* **21** 467